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ELEMENTARY PROOFS OF AN INEQUALITY FOR SYMMETRIC FUNCTIONS FOR --ETC(U)
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ELEMENTARY PROOFS OF AN INEQUALITY FOR SYMMETRIC FUNCTIONS FOR $n \leq 5$

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ABSTRACT

For $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ let the elementary symmetric functions $\psi_j = \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\psi_j(x) = \sum_{1\le i_1<\ldots<i_j\le n} x_{i_1}\ldots x_{i_j}, \quad j = 1,\ldots,n.$$

So the real polynomial p_x of degree n with leading coefficient 1 and zeros in $-x_1,\ldots,-x_n$ is given by $p_x(t) = t^n + \sum_{i=1}^{n} \psi_i(x)t^{n-i}$.

Let $x,y \in \mathbb{R}_+^n$ be points with $\psi_i(x) \le \psi_i(y)$ for $i = 1,\ldots,n$.

It was conjectured (see [2]) that this implies $\psi_i(x^\alpha) \le \psi_i(y^\alpha)$ for every $\alpha \in (0,1]$ and $i = 1,\ldots,n$, where x^α is defined by $x^\alpha = (x_1^\alpha,\ldots,x_n^\alpha)$.

By an argument involving total positivity, this conjecture may be reduced to the problem of finding a piecewise differentiable path $\{\phi(t)\,|\,t \in [0,1]\}$ in $\mathbb{R}_+^n$ with $\phi(0) = x$, $\phi(1) = y$ and such that $\psi_i(\phi(t))$ is monotone increasing with t for each $i = 1,\ldots,n$.

This problem looks deceivingly simple but was only recently solved by Efroymson, Swartz and Wendroff using a rather involved argument.

We give elementary proofs for $n \le 5$.

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SIGNIFICANCE AND EXPLANATION

Some aspects of the heat transfer in the emergency cooling of nuclear reactors lead to a nonlinear eigenvalue problem, the so-called model quelch front problem. Laquer and Wendroff suggested a procedure for computing bounds of the eigenvalue which depend - among other things - on the validity of a certain inequality for elementary symmetric functions. This inequality is of interest in itself and was recently proved by Efroymson, Swartz and Wendroff using a fairly complicated argument. We give an elementary proof for $n \leq 5$.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ELEMENTARY PROOFS OF AN INEQUALITY FOR SYMMETRIC FUNCTIONS FOR $n \leq 5$

Roland Zielke

Let $\mathbb{R}^n_{\substack{+\\(-)}} = \{z \in \mathbb{R}^n \mid \bigwedge_i z_i \underset{(\leq)}{\geq} 0\}$ and $\Delta^n_{\substack{+\\(-)}} = \{z \in \mathbb{R}^n_{\substack{+\\(-)}} \mid z_1 \underset{(\geq)}{\leq} z_2 \underset{(\geq)}{\leq} \ldots \underset{(\geq)}{\leq} z_n\}$.

Let $\sigma : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma: x \to \sigma(x)$, be defined by

$$\prod_{i=1}^{n} (t-x_i) = t^n + \sum_{i=1}^{n} \sigma_i(x) t^{n-i} =: p_x(t) \text{ for } t \in \mathbb{R}.$$

So we have $\sigma_i(x) = \sum_{1 \leq j_1 < \ldots < j_i \leq n} (-1)^i x_{j_1} \ldots x_{j_i}, i = 1, \ldots, n,$

and $\sigma(\mathbb{R}^n_-) \subset \mathbb{R}^n_+$.

Let $\mathbb{R}^n$ be partially ordered by "$x < y$ iff $x_i \leq y_i$ for $i = 1, \ldots, n$ and $x \neq y$". Let $x, y \in \Delta^n_-$ be points with $\sigma(x) < \sigma(y)$ and $M = \{z \in \Delta^n_- \mid \sigma(x) \leq \sigma(z) \leq \sigma(y)\}$. So M is compact.

<u>Theorem A:</u> a) There is a continuous mapping $\phi: [0,1] \to M$ with $\phi(0) = x$, $\phi(1) = y$ and $\sigma(\phi(u)) < \sigma(\phi(v))$ for all $u, v \in [0,1]$ with $u < v$.

b) ϕ is continuously differentiable except on a finite set.

By an argument involving total positivity (see [1]) one may derive from theorem A the following result:

<u>Theorem B:</u> If z^α is defined by $z^\alpha = (-|z_1|^\alpha, \ldots, -|z_n|^\alpha)$ for $z \in \Delta^n_-$ and $\alpha \in \mathbb{R}$, we have $\sigma(x^\beta) \leq \sigma(y^\beta)$ for $\beta \in (0,1]$.

Subsequently we shall prove theorem A for $n \leq 5$.

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Proof: a) It is sufficient to find a $\delta>0$ and a $g \in \mathbb{P}_{n-1}$ with nonnegative coefficients such that $p_x + \lambda g$ has n nonpositive real zeros $x_1^{(\lambda)},\dots,x_n^{(\lambda)}$ for all $\lambda \in [0,\delta]$ and $\sigma(x^{(\lambda)})$ is strictly increasing for $\lambda \in [0,\delta]$.

For $x_n<x_{n-1}<\dots<x_1$ the claim is trivial. Also trivial is the following

Lemma 1: If $y<x$ and $y_1<0$, then $\sigma_i(x) < \sigma_i(y)$ for all i.

We denote $d := p_y - p_x$. We consider the cases n = 2,3,4,5 seperately:

n = 2:

$x_2 = x_1$: choose $g(t) = t$, if $\sigma_1(x) < \sigma_1(y)$.

If $\sigma_1(x) = \sigma_1(y)$, we have $d(t) = \alpha$ for some $\alpha>0$, and p_y has no zeros , a contradiction.

n = 3:

case 1: $x_3 < x_2 = x_1$: choose $g(t) = t$, if $\sigma_1(x) < \sigma_1(y)$.

Otherwise we have $d(t) = \alpha + \beta t^2$ for some $\alpha,\beta \in \mathbb{R}_+$, and lemma 1 gives a contradiction.

case 2: $x_3 = x_2 < x_1$: choose $g(t) = 1$, if $\sigma_0(x) < \sigma_0(y)$;
choose $g(t) = t^2$ if $\sigma_2(x) < \sigma_2(y)$.

Otherwise we have $d(t) = \alpha t$ for some $\alpha \in \mathbb{R}_+$, implying $x_i < y_i$ for i = 1,2,3, a contradiction.

case 3: $x_1 = x_2 = x_3$: choose $g(t) = t(t-x_1)$, if $\sigma_i(x) < \sigma_i(y)$ for i = 1,2.

Otherwise, if $\sigma_1(x) = \sigma_1(y)$, go to n=3, case 1.

If $\sigma_2(x) = \sigma_2(y)$, consider p_x', p_y' and go to n=2.

n = 4:

case 1: $x_4 < x_3 = x_2 < x_1$: choose $g(t) = 1$, if $\sigma_0(x) < \sigma_0(y)$;
choose $g(t) = t^2$, if $\sigma_2(x) < \sigma_2(y)$.

Otherwise we have $d(t) = \alpha t + \beta t^3$ for some $\alpha, \beta \in \mathbb{R}_+$.

$$\Rightarrow d'(t) = \alpha + 3\beta t^2 \Rightarrow p_y(t) > p_x(t) \text{ and}$$

$p_y'(t) > p_x'(t)$ for $t \in (-\infty, 0)$.

So all zeros of p_y' are smaller than all zeros of p_x', yielding $\sigma_2(x) < \sigma_2(y)$, a contradiction.

<u>case 2</u>: a) $x_4 < x_3 < x_2 = x_1$ or b) $x_4 = x_3 < x_2 < x_1$:

choose $g(t) = t$, if $\sigma_1(x) < \sigma_1(y)$;

choose $g(t) = t^3$, if $\sigma_3(x) < \sigma_3(y)$;

otherwise we have $d(t) = \alpha + \beta t^2$, so $d > 0$, $d' < 0$, $d'' > 0$ on $(-\infty, o)$ and $d'(0) = 0$.

For a) this implies $Z(p_y') \subset (-\infty, x_3)$, but also $Z(p_y') \cap (x_1, 0) \neq \phi$, a contradiction.

For b) this implies that either all zeros of p_y' are larger than all zeros of p_x' , or that all zeros of p_y'' are larger than all zeros of p_x'', in both cases a contradiction.

<u>case 3:</u> $x_4 < x_3 = x_2 = x_1$: choose $g(t) = t(t-x_1)$, if $\sigma_1(x) < \sigma_1(y)$ and $\sigma_2(x) < \sigma_2(y)$.

Otherwise, if $\sigma_2(x) = \sigma_2(y)$, we have $d(t) = \alpha + \beta t + \gamma t^3$, so $d'(t) = \beta + 3\gamma t^2$. Now go to <u>n=3, case 1</u>.

If $\sigma_1(x) = \sigma_1(y)$, we have $d(t) = \alpha + \beta t^2 + \gamma t^3$. So d has only one zero z in $(-\infty, 0)$, d' has only one zero z' in $(-\infty, 0)$, $d'(0) = 0$, $z \le z' \le 0$.

If p_1 has no zero in $(x_1, 0)$, the same holds for p_y'. But then all zeros of p_y' are smaller than all zeros of p_x', and lemma 1 gives a contradiction.

If p_y has a zero in $(x_1, 0)$ we have $z \in (x_1, 0)$ and thus $p_x < p_y$ and $p_x' > p_y'$ on $(-\infty, z)$. But then again the zeros of p_y' are smaller than those of p_x'.

case 4: $x_4 = x_3 = x_2 < x_1$: choose $g(t) = t - x_2$, if $\sigma_o(x) < \sigma_o(y)$ and $\sigma_1(x) < \sigma_1(y)$; choose $g(t) = t^2(t-x_2)$, if $\sigma_2(x) < \sigma_2(y)$ and $\sigma_3(x) < \sigma_3(y)$.

Otherwise: a) If $\sigma_o(x) = \sigma_o(y)$ and $\sigma_2(x) = \sigma_2(y)$, go to n = 4, case 1.

b) If $\sigma_1(x) = \sigma_1(y)$ and $\sigma_3(x) = \sigma_3(y)$, go to n = 4, case 2b.

c) If $\sigma_o(x) = \sigma_o(y)$ and $\sigma_3(x) = \sigma_3(y)$, we have $d(t) = \alpha t + \beta t^2$ and $\alpha, \beta > 0$ w.l.o.g.. So p_1'' has its zeros in (x_2, x_1), and d has its negative zero in (x_1, o). But then $x_i \le y_i$ for all i, a contratiction.

d) If $\sigma_1(x) = \sigma_1(y)$ and $\sigma_2(x) = \sigma_2(y)$, we have $d(t) = \alpha + \beta t^3$ and $\alpha, \beta > 0$ w.l.o.g. If d had its zero z in $(-\infty, x_1]$, we would have $y_i \le x_i$ for all i in contradiction to lemma 1. => $z \in (x_1, 0)$. Let z_1 be the local minimum of p_x. We have $p_y' > 0$ in $[z_1, 0]$, so $Z(p_y') \subset (-\infty, x_2)$, for otherwise p_y would have two local extrema in (x_2, z) with no zero in between. But this again yields a contradiction to lemma 1.

case 5: $x_4 = x_3 = x_2 = x_1$: choose $g(t) = t(t-x_1)^2$, if $\sigma_i(x) < \sigma_i(y)$ for i = 1,2,3.

Otherwise, if $\sigma_i(x) = \sigma_i(y)$ for i = 2 or i = 3, consider p_x' and p_y', i.e., go to n=3, case 3.

If $\sigma_1(x) = \sigma_1(y)$, we have $d(t) = \alpha + \beta t^2 + \gamma t^3$ with $\beta, \gamma > 0$ w.l.o.g. Go to n=4, case 3, corresponding case.

n = 5:

We use the following notations:

The zeros of p_x' are z_4, z_3, z_2, z_1 with $z_4 \le z_3 \le z_2 \le z_1$.

The zeros of p_x'' are w_3, w_2, w_1, with $w_3 \le w_2 \le w_1$.

The negative zeros of d are p,q,r,... with p≤q≤r≤...

The negative zeros of d' are p',q',r' with p'≤q'≤r'

The negative zeros of d" are p",q" with p"≤q".

The statement "$\sigma_i(x) < \sigma_i(y)$" is called A_i, i = 0,1,...,4.

α, β, γ, δ are nonnegative real numbers.

case 1:

a) $x_5 < x_4 < x_3 < x_2 = x_1$

b) $x_5 < x_4 = x_3 < x_2 < x_1$

c) $x_5 < x_4 = x_3 < x_2 = x_1$

Choose $g(t) = t$ if A_1, choose $g(t) = t^3$ if A_3.

If $\neg A_1 \wedge \neg A_3$, we have $d(t) = \alpha + \beta t^2 + \gamma t^4 \Rightarrow d'(o) = 0 \wedge d' < 0 < d''$ on $(-\infty, 0)$.

a) We have $Z(p_y) \subset (-\infty, x_5) \cup (x_4, x_3)$ and $Z(p_y') \cap (x_1, 0) \neq \phi \Rightarrow Z(p_y) \cap (x_1, 0) \neq \phi$, contradiction.

c) Follows from a).

b) We have $Z(p_y) \subset (-\infty, x_5) \cup (x_2, x_1)$ and $Z(p_y') \subset (-\infty, z_4) \cup (x_3, z_2) \cup (z_1, 0)$.

If $Z(p_y'') \subset (-\infty, w_3]$, lemma 1 gives a contradiction to $\neg A_3$.

$\Rightarrow \#(Z(p_y'') \cap (-\infty, w_3]) = 1 \wedge \#(Z(p_y'') \cap [w_2, w_1]) = 2$

$\Rightarrow p_y'$ has 2 zeros in (z_3, z_1)

$\Rightarrow p_y'$ has 2 zeros in (z_3, z_2)

$\Rightarrow p_y$ has 1 zeros in $(z_3, z_2) \subset (x_3, x_2)$, contradiction.

case 2:

a) $x_5 < x_4 < x_3 = x_2 < x_1$

b) $x_5 = x_4 < x_3 < x_2 < x_1$

c) $x_5 = x_4 < x_3 = x_2 < x_1$

Choose $g(t) = 1$ if A_o,

$g(t) = t^2$ if A_2,

$g(t) = t^4$ if A_4.

If $\neg A_o \wedge \neg A_2 \wedge \neg A_4$, we have $d(t) = \alpha t + \beta t^3 \Rightarrow d' > 0 > d''$ on $(-\infty, 0)$.

a) p_y has one zero in $(x_1, 0)$ and 4 zeros in $(x_5, x_4) \wedge Z(p_y') \subset (z_4, z_3)$

$\Rightarrow Z(p_y'') \subset (z_4, z_3)$. But $p_y''(0) = p_x''(0) \geq 0 \wedge p_y''(w_1) < p_x''(w_1) = 0 \Rightarrow$ $Z(p_y'') \cap (w_1 0) \neq \phi$, contradiction.

b) p_y has one zero in $(x_1, 0)$ and 4 zeros in $(x_3, x_2) \wedge Z(p_y') \subset (z_2, z_1)$

$\Rightarrow Z(p_y'') \subset (w_1, z_1)$, contradiction.

case 3:

a) $x_5 \le x_4 < x_3 = x_2 = x_1$

b) $x_5 = x_4 = x_3 < x_2 \le x_1$

a) Choose $g(t) = t(t-x_1)$ if $A_1 \wedge A_2$,

$g(t) = t^3(t-x_1)$ if $A_3 \wedge A_4$,

$g(t) = t(t^3-x_1^3)$ if $A_1 \wedge A_4$. Otherwise we have:

a) 1) $\neg A_2 \wedge \neg A_4$: consider p_x', p_y' and go to n=4, case 2a).

a) 2) $\neg A_1 \wedge \neg A_3$: we have $d(t) = \alpha + \beta t^2 + \gamma t^4 \wedge d' < 0$ on $(-\infty,0) = Z(p_y') \cap (x_1,0) \neq \emptyset$. But $Z(p_y) \subset (-\infty,x_1) \Rightarrow Z(p_y') \subset (-\infty,x_1)$ contradiction.

a) 3) $\neg A_1 \wedge \neg A_4$: we have $d(t) = \alpha + \beta t^2 + \gamma t^3$ and $\beta > 0$ w.l.o.g. So d, d', d'' have exactly one negative zero each, and $p < p' < p''$.

$$\left.\begin{array}{l} \text{If } p \in [x_1,0] \Rightarrow Z(p_y) \subset (x_5,x_4) \cup (x_1,p) \Rightarrow Z(p_y') \subset (z_4,z_3) \\ \text{If } p < x_1 \Rightarrow Z(p_y) \subset (-\infty,x_1) \Rightarrow Z(p_y') \subset (-\infty,x_1) \Rightarrow \\ \quad p' \in (x_1,0) \Rightarrow Z(p_y') \subset (z_4,z_3) \end{array}\right\} \Rightarrow$$

$Z(p_y'') \subset (z_4,z_3)$. But $p'' \in (x_1,0) \Rightarrow p_y''(x_1) < 0 \Rightarrow Z(p_y'') \cap (x_1,0) \neq \emptyset$, contradiction.

b) Choose $g(t) = t(t-x_5)$ if $A_1 \wedge A_2$,

$g(t) = t^3(t-x_5)$ if $A_3 \wedge A_4$,

$g(t) = t(t^3-x_5^3)$ if $A_1 \wedge A_4$. Otherwise we have:

b) 1) $\neg A_2 \wedge \neg A_4$: consider p_x', p_y' and go to n=4, case 2b).

b) 2) $\neg A_1 \wedge \neg A_3$: we have $d(t) = \alpha + \beta t^2 + \delta t^4$, so $d' < 0 < d''$ on $(-\infty,0)$ and

$$Z(p_y) \subset (-\infty,x_5) \cup (x_2,x_1),$$
$$Z(p_y') \subset (-\infty,z_2) \cup (z_1,0),$$
$$Z(p_y'') \subset (-\infty,x_5) \cup (w_2,w_1).$$

p_y'' has one zero in $(-\infty,x_5)$ and two zeros in (w_2,w_1), for otherwise lemma 1 and A_3 give a contradiction.

b) 2) a) $Z(p_y) \subset (-\infty,x_5) \Rightarrow$ lemma 1 contradicts $\neg A_3$.

b) 2) b) $\#(Z(p_y)\cap(x_2,x_1) = 2 \Rightarrow \#(Z(p_y')\cap(z_1,0) = 1$

$\Rightarrow p_1'$ has 3 zeros in $(-\infty,x_5)$

$\Rightarrow p_y''$ has 2 zeros in $(-\infty,x_5)$, contradiction.

b) 2) c) $\#(Z(p_y)\cap(x_2,x_1)) = 4 = \#(Z(p_y')\cap(z_1,x_1)) = 3$

$\Rightarrow \#(Z(p_y'')\cap(z_1,x_1)) = 2$, contradiction

b) 3) $\neg A_1 \wedge \neg A_4$: we have $d(t) = \alpha + \beta t^2 + \gamma t^3$.

b) 3) a) $p' \in (z_1,0)$: If $\#(Z(p_y')\cap[z_1,0)) = 2$, lemma 1 and $\neg A_4$ give a contradiction.

If $Z(p_y')\subset(-\infty,z_1) \Rightarrow Z(p_y')\subset(z_2,z_1]$. From $d''<0$ in $(-\infty,p')$ follows $Z(p_y'')\subset(w_1,z_y)$, contradiction.

b) 3) b) $p' \in [z_2,z_1) \Rightarrow$ lemma 1 and $\neg A_4$ give a contradiction.

b) 3) c) $p' \in [x_5,z_2) \Rightarrow Z(p_y')\subset(p',z_2)\cup(z_1,0)$

$\Rightarrow \#(Z(p_y')\cap(p',z_3)) = 3 \Rightarrow \#(Z(p_y)\cap(p',z_2)) = 2$.

But $p_y>p_x>0$ in $(p,z_2)\cap(x_5,z_2)\supset(p',z_2)$, contradiction.

b) 3) d) $p'<x_5$: If $\#(Z(p_y')\cap(x_5,z_2)) = 2 \Rightarrow Z(p_y)\cap(x_5,z_2) \neq \phi$, but $p_y>p_x>0$ on (x_5,z_2), contradiction.

<u>case 4:</u>

$x_5 < x_4 = x_3 = x_2 < x_1$: Choose $g(t) = t-x_3$ if $A_0 \wedge A_1$,

$g(t) = t^2(t-x_3)$ if $A_2 \wedge A_3$,

$g(t) = t^3-x_3^3$ if $A_0 \wedge A_3$. Otherwise

we have:

1) $\neg A_1 \wedge \neg A_3$: consider p_x', p_y' and go to <u>n=4, case 1</u>.

2) $\neg A_0 \wedge \neg A_2$: we have $d(t) = \alpha t + \beta t^3 + \gamma t^4$.

$\Rightarrow d, d', d''$, have each exactly one negative zero, and $p<p'<p''$.

One checks that $Z(p_y'')\subset(x_3,0)$ or $Z(p_y'')\subset(-\infty,x_3]$ are impossible.

a) $p' \in [z_1,0) \Rightarrow Z(p_y')\subset(-\infty,z_4)\cup(z_1,0)$.

If p_y' had 3 zeros in (z_1,p'), p_y'' would have 2 zeros in (z_1,p'), contradiction.

If p_y' had 3 zeros in $(-\infty, z_4)$, p_y'' would have 2 zeros in $(-\infty, z_4)$

$\Rightarrow Z(p_y'') \subset (-\infty, w_3) \Rightarrow \neg A_2$ and lemma 1 give a contradiction.

b) $p' \in [x_3, z_1)$:

If p_y' has exactly one zero $> p'$, p_y'' has at most one zero $\geq x_3$.

So p_y'' has no zero $\geq x_3$, contradiction.

If p_y' has 3 zeros in $(p', 0)$, p_y has 4 zeros in $(x_1, 0)$.

$\Rightarrow Z(p_y) \subset (x_1, 0)$, contradiction.

c) $p' \in (-\infty, x_3) \Rightarrow \#(Z(p_y') \cap (x_3, z_1)) \geq 2$, for otherwise $Z(p_y'') \subset (-\infty, x_3)$, contradiction. So we have $\#(Z(p_y) \cap (x_3, z_1)) \geq 1$, but this contradicts $p_y < p_x < 0$ in (x_3, z_1).

3) $\neg A_0 \wedge \neg A_3$: we have: $d(t) = \alpha t + \beta t^2 + \gamma t^4$. Then d and d' have exactly one negative zero each, $p < p'$, and $d'' > 0$ on $(-\infty, 0]$ w.l.o.g..

So p_y'' has 2 zeros in (x_3, w_1), i.e., p_y' has a local maximum r and a local minimum s with $p' < r < s < w_1$.

$\left.\begin{array}{l}\Rightarrow p_y \text{ has a local maximum } l \in (r,s), \text{ and } d(l) > 0. \\ \text{For } \varepsilon > 0 \text{ sufficiently small, } d(-\varepsilon) < 0\end{array}\right\} \Rightarrow p \in (l, 0) \Rightarrow p' < p$, contradiction.

<u>case 5</u>:

$x_5 = x_4 < x_3 < x_2 = x_1$: choose $g(t) = t(t-x_3)$ if $A_1 \wedge A_2$,

$g(t) = t^3(t-x_3)$ if $A_3 \wedge A_4$,

$g(t) = t(t^3-x_3)$ if $A_1 \wedge A_4$. Otherwise we have:

1) $\neg A_1 \wedge \neg A_3$: go to <u>n=5, case 1a</u>.

2) $\neg A_2 \wedge \neg A_4$: we have $d(t) = \alpha + \beta t + \gamma t^3$, so $d' > 0 > d''$ on $(-\infty, 0)$.

$\Rightarrow Z(p_y') \subset (x_5, z_3) \cup (z_2, x_1) \wedge Z(p_y'') \subset (w_3, w_2) \cup (w_1, 0)$

$\Rightarrow p_y'$ has at least 2 zeros in (z_2, x_1) and 1 zero in (x_5, w_3)

$\Rightarrow p_y$ has a local minimum in (z_2, x_1) and a local maximum in (x_5, w_3).

$\Rightarrow d$ has at least 2 zeros in $(-\infty, 0)$, contradiction.

3) $\neg A_1 \wedge \neg A_4$: we have $d(t) = \alpha + \beta t^2 + \gamma t^3$.

$\Rightarrow$ d, d', d'' have each exactly one negative zero, and $p < p' < p''$.

We have either $Z(p_y) \subset (-\infty, x_3]$ or $Z(p_y) \subset (x_3, 0)$:

$Z(p_y) \subset (-\infty, x_3]$ implies $Z(p_y'') \subset (-\infty, x_3)$ $\Rightarrow$ $p' > x_1$ $\Rightarrow$ $p'' > x_1$ contradiction.

$Z(p_y) \subset (x_3, 0)$ implies $Z(p_y) \subset (z_2, 0)$ $\Rightarrow$ $Z(p_y'') \subset (z_2, 0)$ $\Rightarrow$ $Z(p_y'') \subset (z_2, w_1)$

$\Rightarrow$ $p'' < w_1$, $\Rightarrow$ $p' < w_1$ $\Rightarrow$ $Z(p_y') \cap (x_1, 0] \neq \emptyset$ $\Rightarrow$ $x_1 < p < p' < w_1$ contradiction.

<u>case 6</u>:

$x_5 = x_4 = x_3 = x_2 < x_1$: choose $g(t) = (t - x_5)^2$ if $A_0 \wedge A_1 \wedge A_2$

$g(t) = t^2 (t - x_5)^2$ if $A_2 \wedge A_3 \wedge A_4$

$g(t) = (t^2 + x_5 t + x_5^2)(t - x_5)^2 =$

$= t^4 - x_5 t^3 - x_5^3 t + x_5^4$, if $A_0 \wedge A_1 \wedge A_3 \wedge A_4$.

Otherwise we have:

1) $(\neg A_1 \wedge \neg A_3)$ or $(\neg A_1 \wedge \neg A_4)$: consider p_x', p_y' and go to <u>n=4, case 4</u>.

2) $\neg A_0 \wedge \neg A_3$: go to <u>n=5, case 4,3)</u>.

3) $\neg A_0 \wedge \neg A_4$: we have $d(t) = \alpha t + \beta t^2 + \gamma t^3$.

$\Rightarrow$ d has 2, d' has 2, d'' has 1 negative zero, and $p < p' < q < q' < 0, p' < p'' < q'$.

From $p'' < w_1$ follows that p_y'' has at least 2 zeros in $(\max\{p_1'' x_5\}, w_1)$.

So p_y' has a local maximum r and a local minimum s with $p'' < r$ and

$x_5 < r < s < w_1$. $\Rightarrow$ $d'(r) > 0$ $\Rightarrow$ either $r' < q' < r$ or $r < p' < q'$.

As p_y has a local maximum $l \in (r,s)$ and $d(l) > 0$, we have $l \in (p,q)$,

so $r < l < q < q'$, and so finally $r < p' < q'$ $\Rightarrow$ $r < p''$, contradiction.

<u>case 7</u>:

$x_5 < x_4 = x_3 = x_2 = x_1$: choose $g(t) = t(t - x_1)^2$ if $A_1 \wedge A_2 \wedge A_3$.

Otherwise we have:

1) $\neg A_2$ or $\neg A_3$: consider p_x', p_y' and go to <u>n=4, case 3</u>.

2) $\neg A_1$: we have $d(t) = \alpha + \beta t^2 + \gamma t^3 + \delta t^4$ with $\beta, \gamma > 0$ w.l.o.g..

$\Rightarrow$ d has at most 2 zeros in $(-\infty,0)$. If d had no zero or one double zero in $(-\infty,0)$, lemma 1 and $\neg A_1$ would give a contradiction.

$\Rightarrow$ d has exactly 2 zeros in $(-\infty,0)$, as well as d' and d", and $p<p'<q<q'<0$ and $p'<p''<q'<q''<0$.

<u>claim 1</u>: $Z(p_y'')\subset(-\infty,w_3] \Rightarrow$ d" has no zero in $[x_1,0]$.

<u>Proof</u>: explicit computation gives $p_x''<p_y''$ on $[x_1,0]$.

<u>claim 2</u>: $x_5\le q \Rightarrow Z(p_y')\subset(-\infty,q')$.

<u>claim 3</u>: $p'\le x_1\le q' \Rightarrow$ either $p'<z_4$, or p_y' has 2 zeros in $(q',0)$.

<u>Proof</u>: If p_y' has less than 2 zeros in $(q',0)$, p_y' has no zero there, so $Z(p_y')\subset(-\infty,x_1]$. If now p_y' had a zero $\le z_4$, lemma 1 and $\neg A_1$ would yield a contradiction.

From $q\le x_5$ would follow A_1 by lemma 1, a contradiction. So we have $x_5\le q \Rightarrow p'<x_1$, for otherwise $x_1<p'<q' \Rightarrow Z(p_y')\subset(x_1,p')$ because of claim 2 $\Rightarrow Z(p_y'')\subset(x_1,p')$, contradiction.

$$\left.\begin{array}{l} \text{a) } x_1<q \Rightarrow x_1<q' \Rightarrow Z(p_y')\subset(-\infty,x_1) \\ \text{b) } q\le x_1 \Rightarrow Z(p_y)\subset(-\infty,x_1) \Rightarrow Z(p_y')\subset(-\infty,x_1) \end{array}\right\} \Rightarrow Z(p_y')\subset(z_4,x_1),\ \text{for}$$

otherwise lemma 1 and $\neg A_1$ yield a contradiction.

$\Rightarrow Z(p_y'')\subset(-\infty,x_1)\wedge x_1<q'<q'' \Rightarrow x_1<p'' \Rightarrow Z(p_y'')\subset(-\infty,w_3)$ contradiction to claim 1.

<u>case 8</u>:

$x_5 = x_4 = x_3 = x_2 = x_1$: choose $g(t) = t(t-x_1)^3$ if $A_1 \wedge A_2 \wedge A_3 \wedge A_4$.

1) $\neg A_2$ or $\neg A_3$ or $\neg A_4$: p_x' and p_y' can be treated as <u>n=4, case 5</u>.

2) $\neg A_1$: same as <u>n=5, case 7</u>.

b) Let $\|\ \|$ denote any fixed norm in $\mathbb{R}^n$.

We construct $f_1,f_2,\ldots,\in \mathbb{P}_n$ with corresponding zeros $x^{(1)},x^{(2)},\ldots,\in M$ as follows: Let $f_o = p_x$ and $x^{(o)} = x$. If for $k\ge 0$, $x^{(k)}$ and f_k are

given, for every

$$g \in S := \{f \in \mathbb{P}_{n-1} ; \max_{t\in[0,1]} |f(t)| = 1\},$$

let δ_g be maximal such that

a) for all $\lambda \in [0,\delta_g]$, $f_k+\lambda g$ has n zeros $z_{g_1}^{(\lambda)}, \ldots, z_{g_n}^{(\lambda)}$ with $z_g^{(\lambda)} \in M$

b) $\sigma(z_g^{(\lambda)})$ is strictly increasing for $\lambda \in [0,\delta_g]$.

Let $\hat{g} \in S$ be a function with

$$\|\sigma(z_{\hat{g}}^{(\delta_{\hat{g}})} - \sigma(f_k)\| = \max_{g\in S} \|\sigma(z_g^{(\delta_g)} - \sigma(f_k)\|,$$

and define $f_{k+1} = f_k + \delta_{\hat{g}}\hat{g}$, $x^{(k+1)} = z_{\hat{g}}^{(\delta_{\hat{g}})}$.

So p_x and every f_k are connected by a path along which σ is strictly increasing, and this path corresponds to a polygonal are in $\sigma(M)$ with corners $\sigma(x^{(o)}, \sigma(x^{(1)}, \ldots, \sigma(x^{(k)})$. We have to show $f_k = p_y$ occurs for some k.

Suppose the contrary, i.e. $\sigma(x^{(k)}) < \sigma(y)$ for all $k = 1,2,\ldots$.

As $\{\sigma(x^{(k)})\}$ is an increasing sequence, $\sigma^\infty := \lim_{k\to\infty} \sigma(x^{(k)})$ exists.

Let $x^\infty := \lim_{k\to\infty} x^{(k)}$, so $\sigma^\infty = \sigma(x^{(\infty)})$, and f_∞ the corresponding polynomial.

There is a $g \in \mathbb{P}_{n-1}$, and a $\delta > 0$ such that $f_\infty+\lambda g$ has n zeros $z_1^{(\lambda)}, \ldots, z_n^{(\lambda)}$ with $Z^{(\lambda)} \in \Delta_-^n$ for all $\lambda \in [0,2\delta]$, and $\sigma(z^{(\lambda)})$ is strictly increasing with $\lambda \in [0,2\delta]$. Let $\varepsilon = \|\sigma(z^{(\delta)}) - \sigma(x^{(\infty)})\|$.

We shall show that for every $\varepsilon > 0$, there is an index and a $\tilde{g} \in \mathbb{P}_{n-1}$ (near g) such that

1) $f_k+\lambda\tilde{g}$ has n zeros $\tilde{z}_1^{(\lambda)}, \ldots, \tilde{z}_n^{(\lambda)}$ with $\tilde{z}^{(\lambda)} \in \Delta_-^n$ for all $\lambda \in [0,\delta]$,

2) $\sigma(\tilde{z}^{(\lambda)})$ is strictly increasing for $\lambda \in [0,\delta]$,

3) $\| \sigma(z^{(\delta)}) - \sigma(\tilde{z}^{(\delta)}) \| < \varepsilon$.

(This implies $\| \sigma(x^{(k+1)}) \| \geq \| \sigma(\tilde{z}^{(\delta)}) \| \geq \| \sigma^{\infty} \| + \alpha - \varepsilon > \| \sigma^{\infty} \|$

for all sufficiently small $\varepsilon > 0$, a contradiction.)

Let $\tilde{\varepsilon} > 0$ be arbitrarily fixed and k so large that

$|(f_{\infty} - f_k)(t)| < \tilde{\varepsilon}$ for all $t \in I := [2x_n^{\infty} - 1, 1]$, *and*

$\| x^{(k)} - x^{\infty} \| < \tilde{\varepsilon}$.

So in an $\tilde{\varepsilon}$-neighbourhood of every zero z of f^{∞} of multiplicity m, f_k has exactly m zeros couting multiplicities.

As the functions g in part a) of the proof were constructed only in view of the multiplicities of the zeros of f^{∞}, $\tilde{g}$ can be constructed correspondingly in view of the zeros of f_k.

As an example, we consider the case <u>n=5, case 8</u> (leaving the analogous details of the other cases to the reader):

For $f_{\infty}(t) = (t - x_1^{\infty})^5$, we had $g(t) = (t - x_1)^3 t$.

For $f_k(t) = \prod_{i=1}^{5} (t - x_i^{(k)})$ with $x_5^{(k)} \leq x_4^{(k)} \leq \ldots \leq x_1^{(k)}$, we choose

$\tilde{g}(t) = (t - x_2^{(k)})(t - x_3^{(k)})(t - x_4^{(k)})t$

$$\Rightarrow \max_{t \in I}\{|(g - \tilde{g})(t)|\} = O(\tilde{\varepsilon}), \text{ and}$$

$$\max_{t \in I}\{|(f_{\infty} + \delta g)(t) - (f_k + \delta \tilde{g})(t)|\} = O(\tilde{\varepsilon}).$$

As $f_{\infty} + \delta g$ has 2 simple zeros $\neq x_1$, $f_k + \delta \tilde{g}$ has simple zeros near these.

For sufficiently small $\tilde{\varepsilon}$ and large k, statement 3) above holds, too.

References

[1] Efroymson, G.A., Swartz, B. and B. wendroff: An inequality for symmetric functions. Submitted to "Advances in Mathematics".

[2] Laquer, H.T. and B. Wendroff: Bounds for the model quelch front; to appear.

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For $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ let the elementary symmetric functions $\psi_j = \mathbb{R}^n \to \mathbb{R}$ be defined by

$\psi_j(x) = \sum_{1 \leq i_1 < \ldots < i_j \leq n} x_{i_1} \ldots x_{i_j}$, $j = 1,\ldots,n$. So the real polynomial p_x of degree n

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ABSTRACT (cont.)

with leading coefficient 1 and zeros in $-x_1,\dots,-x_n$ is given by

$p_x(t) = t^n + \sum_{i=1}^{n} \psi_i(x)t^{n-i}$.

Let $x,y \in \mathbb{R}_+^n$ be points with $\psi_i(x) \leq \psi_i(y)$ for $i = 1,\dots,n$. It was conjectured (see [2]) that this implies $\psi_i(x^\alpha) \leq \psi_i(y^\alpha)$ for every $\alpha \in (0,1]$ and $i = 1,\dots,n$, where x^α is defined by $x^\alpha = (x_1^\alpha,\dots,x_n^\alpha)$. By an argument involving total positivity, this conjecture may be reduced to the problem of finding a piecewise differentiable path $\{\phi(t)|t \in [0,1]\}$ in $\mathbb{R}_+^n$ with $\phi(0) = x$, $\phi(1) = y$ and such that $\psi_i(\phi(t))$ is monotone increasing with t for each $i = 1,\dots,n$ (see [1]). This problem looks deceivingly simple but was only recently solved by Efroymson, Swartz and Wendroff using a rather involved argument. We give elementary proofs for $n \leq 5$.